Polylogarithms and Riemann's ζ function

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Riemann's ζ function has been important in statistical mechanics for many years, especially for the understanding of Bose-Einstein condensation. Polylogarithms can yield values of Riemann's ζ function in a special limit. Recently these polylogarithm functions have unified the statistical mechanics of ideal gases. Our particular concern is obtaining the values of Riemann's ζ function of negative order suggested by a physical application of polylogs. We find that there is an elementary way of obtaining them, which also provides an insight into the nature of the values of Riemann's ζ function. It relies on two properties of polylogs—the recurrence and duplication relations. The relevance of the limit process in the statistical thermodynamics is described. [S1063-651X(97)01510-9]

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I. INTRODUCTION

Riemann's ζ function perhaps first appeared in statistical mechanics in 1900 in Planck's theory of the blackbody radiation and then in 1912 in Debye's theory of the specific heats of solids [1]. Subsequently, this function has played an important role in the statistical theory of the ideal Bose gas, especially for the understanding of Bose-Einstein condensation (BEC) [2]. More recently, this function together with the Mellin transform has become a powerful tool for the analysis of the thermodynamic potentials [3,4]. It would be no surprise to find fruitful applications of Riemann's ζ function in other areas of today's theoretical physics [5].

Recently it was found that the statistical thermodynamics of ideal gases can be given a unified picture through polylogs defined in terms of the fugacity z and dimensions d [6]. There is richness that this unified picture reveals, such as the anomalous physics in null dimension [7], the Fermi-Bose reflection in $d \geqslant 3$ [8], and the Fermi-Bose equivalence in $d=2$ [9]. These physical results are consequences of some special properties of polylogs. It has been long known that a polylog of integral order becomes Riemann's ζ function of the same order when its argument attains unity [10]. Thus Riemann's ζ function can enter into the unified theory of the statistical thermodynamics via the polylogs quite naturally. Interestingly, we find that this formulation shows another way of evaluating Riemann's ζ function, which is presented in this work.

The classical theory of polylogs begins with Euler's dilog and Landen's trilog, and extends to higher order polylogs such as the quadrilog. In physical applications the order of a polylog is related to physical dimensions d. Thus polylogs of integral order lower than the dilog have been conceived, such as the nil-log and the monolog for the physics in $d=0$ and 2 [6]. There have been suggestions that negative dimensions can be of theoretical interest [11]. They require the polylogs of still lower orders than the nil-log, departing from the direction of the classical theory of polylogs. What we find is that in these circumstances there exists even a simpler relationship between the polylogs and Riemann's ζ function. We can use this relationship to evaluate Riemann's ζ function very simply, perhaps more simply than by most standard methods. It also lends an interesting insight into the nature of the values of Riemann's ζ function.

II. POLYLOGS AND THEIR PROPERTIES

To show their relationship to Riemann's ζ function, we shall introduce a convenient integral representation for polylogs $\mathrm{Li}_s(z)$ of complex numbers s and z [6], defined by

$$\mathrm{Li}_s(z)=\frac{z}{\Gamma(s)}\int_0^1[\log(1/t)]^{s-1}\frac{dt}{1-zt},\quad \mathrm{Im}\ t=0, \tag{1}$$

whenever this integral converges, i.e., $\mathrm{Re}s>0$, $\mathrm{Re}z<1$, and elsewhere by analytic continuation. It is understood that $\log(1/t)$ has its principal value. Evidently there is a branch cut from $z=1$ to ∞. Also if $s=2$, the standard expression for the dilog is recovered [10]. The above equation (1) bears resemblance to an integral representation for Riemann's ζ function $\zeta(s)$ of a complex number s [12].

We shall now state a few useful properties for our purposes which follow directly from Eq. (1).

(a) $\mathrm{Li}_s(z=1)=\zeta(s)$.

(b) If $s=n=2,3,4,\ldots$, $\mathrm{Li}_n(z)$ are classical polylogs, known, respectively, as the dilog, trilog, quadrilog, etc. (Throughout this work we shall reserve n to denote real integers, both positive and negative.)

(c) $\lim_{z\to 0} z^{-1}\,\mathrm{Li}_s(z)=1$. There is a trivial fixed point at the origin.

(d) Recurrence relation

$$z\frac{d}{dz}\mathrm{Li}_{s+1}(z)=\mathrm{Li}_s(z).$$

(e) Duplication relation

$$\mathrm{Li}_s(z)+\mathrm{Li}_s(-z)=2^{1-s}\mathrm{Li}_s(z^2).$$

If $s=n=2$ and 3, one recovers Euler's formula for the dilog and Landen's for the trilog, respectively [10].

(f) $|\mathrm{Li}_s(z=-1)|<\infty$ since the function is analytic at $z=-1$.

(g) If $|z|<1$ and $s=n\geqslant 1$ (also $|z|=1$ included if $n\geqslant 2$),

$$\mathrm{Li}_n(z)=\sum_{k=1}^{\infty} z^k/k^n.$$

(h)

$$\mathrm{Li}_n(1)=\begin{cases}\zeta(n) & \text{if } >1\\ \infty & \text{if } n\leqslant 1.\end{cases}$$

The property (h) is the basis for the existence of BEC if $d\geqslant 3$ and for the absence if $d\leqslant 2$ [1]. It should be noted that if $s=n>1$, there is thus no difference in $\mathrm{Li}_n(1)$ between the one given in (h) and the other given in (a). If $n<1$, however, by $\mathrm{Li}_n(1)$ we shall mean simply the integral (1), which then becomes undefined as shown above because of the singularity in $\mathrm{Li}_n(z)$ at $z=1$. However, one can obtain $\mathrm{Li}_s(1)$, $s=-m$, $m>0$, by analytic continuation in the manner of $\zeta(s=-m<0)$. To denote this latter case— and to avoid the possible confusion we shall use $\zeta(s)$ in place of $\mathrm{Li}_s(1)$ whenever analytic continuation is implied. Since $z=-1$ is not a singular point of $\mathrm{Li}_s(z)$, this kind of distinction need not be made for $\mathrm{Li}_s(-1)$.

III. POLYLOGS OF NEGATIVE INTEGRAL ORDER: POLYPSEUDOLOGS

If $z=1$ in the duplication relation (e), we obtain together with (a),

$$\zeta(s)=(2^{1-s}-1)^{-1}\mathrm{Li}_s(-1). \tag{2}$$

Thus it is possible to evaluate $\zeta(s)$ if $\mathrm{Li}_s(z)$ can be given at $z=-1$. If $s=n=2$, for example, we recover the familiar result $\zeta(2)=\pi^2/6$, given that $\mathrm{Li}_2(-1)=-\pi^2/12$. See the Appendix for $\mathrm{Li}_n(-1)$, $n=2,4,6,\ldots$, obtained purely from the inversion property of classical polylogs. Hence it is elementary to evaluate $\zeta(s=n)$ if $n=2,4,6,\ldots$. To evaluate $\zeta(s=n)$ when $n<1$, we need to know first the form of $\mathrm{Li}_s(z)$ for $s=n=1,0,-1,-2$, etc. These lower order polylogs can be obtained from any one of the higher order ones through repeated differentiation. See the recurrence relation (d). Hence it is sufficient to know one polylog function in closed form. By setting $s=n=1$ in Eq. (1), we immediately obtain the form for the monolog,

$$\mathrm{Li}_1(z)=-\log(1-z), \quad z\neq 1. \tag{3}$$

By applying the recurrence relation to the above form of the monolog and repeating it over and again, we have obtained the lower order polylogs given below to order $n=-8$:

$$\mathrm{Li}_0(z)=z/(1-z)=-1/2, \tag{4}$$

$$\mathrm{Li}_{-1}(z)=z/(1-z)^2=-1/4,$$

$$\mathrm{Li}_{-2}(z)=z(1+z)/(1-z)^3=0,$$

$$\mathrm{Li}_{-3}(z)=z(1+4z+z^2)/(1-z)^4=1/8,$$

$$\mathrm{Li}_{-4}(z)=z(1+z)(1+10z+z^2)/(1-z)^5=0,$$

$$\mathrm{Li}_{-5}(z)=z(1+26z+66z^2+26z^3+z^4)/(1-z)^6=-1/4,$$

$$\mathrm{Li}_{-6}(z)=z(1+z)(1+56z+246z^2+56z^3+z^4)/(1-z)^7=0,$$

$$\begin{aligned}\mathrm{Li}_{-7}(z)&=z(1+120z+1191z^2+2416z^3\\&\quad+1191z^4+120z^5+z^6)/(1-z)^8\\&=17/16,\end{aligned}$$

$$\begin{aligned}\mathrm{Li}_{-8}(z)&=z(1+z)(1+246z+4047z^2+11572z^3+4047z^4\\&\quad+246z^5+z^6)/(1-z)^9\\&=0.\end{aligned}$$

The numerical values given on the right-hand side of Eq. (4) are the polylogs evaluated at $z=-1$. There seems to be no general closed form expression recognizable from these lower order polylogs. One can, however, obtain the polylog of any desired lower order. Thus our results may perhaps be considered all but complete. These polylogs of order $n=0,-1,-2,\ldots$ have shed log character. We might call them polypseudologs, e.g., $\mathrm{Li}_{-2}(z)$ the dipseudolog. This distinction will be found useful.

We can easily verify that these polypseudologs satisfy several important functional properties of polylogs stated in Sec. II (i) The recurrence relation (d) is satisfied. (ii) The duplication relation (e) is satisfied. (iii) They are finite at $z=-1$. See (f). (iv) If $|z|<1$, the expansions of polypseudologs are also given by (g). Hence the condition $s=n\geqslant 1$ given therein may be relaxed to $s=n$. In addition, the polypseudologs $\mathrm{Li}_{-n}(z)$, $n>0$ show the following properties. (v) They have a pole of order $n+1$ at $z=1$. See (h). (vi) They are factorable by z [see (c)] and also by $(z+1)$ if n is an even number. (vii) The numerical coefficients add up to $n!$. (viii) Evidently $\mathrm{Li}_{-n}(-1)=0$ if $n=2,4,6,\ldots$, and some numbers relatable to Bernoulli's numbers if $n=1,3,5,\ldots$. These results may be used to obtain the values of Riemann's ζ function $\zeta(s=n)$, $n<1$ to any desired lower order.

IV. REIMANN'S ζ FUNCTION

We shall first consider one or two special cases. If $s\to 1$ in Eq. (2),

$$\zeta(s\to 1)=\lim_{s\to 1}(2^{1-s}-1)^{-1}\mathrm{Li}_1(-1). \tag{5}$$

From Eq. (3), $\mathrm{Li}_1(-1)=-\log 2$. Also, $(2^{1-s}-1)\to -(s-1)\log 2$ as $s\to 1$. Hence $\zeta(s\to 1)=1/(s-1)$, $s\to 1$, a well-known result. If $s=\frac{1}{2}$ in Eq. (2),

$$\zeta(\tfrac{1}{2})=(\sqrt{2}+1)\mathrm{Li}_{1/2}(-1)=-1.460\ldots, \tag{6}$$

where by (g) $\mathrm{Li}_{1/2}(-1)=-1+1/\sqrt{2}-1/\sqrt{3}+\ldots=-0.6048\ldots$, a slow but converging series [13,14]. This results, Eq. (6), cannot be obtained by the reflection formula of Riemann [12].

From Eq. (2) we see that the coefficient standing before $\mathrm{Li}_s(-1)$ is always finite if s is a negative real number. Hence from Eq. (4) we obtain at once that

$$\zeta(-2m)=0, \quad m=1,2,\ldots. \tag{7}$$

For the others we can evaluate one by one. By using Eq. (4) in Eq. (2), we obtain at once $\zeta(0)=-\frac{1}{2}$, $\zeta(-1)=-\frac{1}{12}$, $\zeta(-3)=\frac{1}{120}$, $\zeta(-5)=-\frac{1}{252}$, $\zeta(-7)=\frac{1}{240}$, etc. According to Titchmarsh's book [15], $\zeta(-2m)=0$ and $\zeta(1-2m)=(-1)^m B_m/(2m)$, $m=1,2,\ldots$, where B_m are the Bernoulli numbers, e.g., $B_1=\frac{1}{6}$, $B_2=\frac{1}{30}$, $B_3=\frac{1}{42}$, $B_4=\frac{1}{30}$, $B_5=\frac{5}{66}$, etc. Insofar as we have determined, we recover the values of Riemann's ζ function very simply.

Observe the remarkable difference between the two classes of numbers, $\zeta(n)$, $n=-1,-3,-5,\ldots$ and $n=2,4,6,\ldots$. The former are rational numbers, whereas the latter are not, containing even powers of π. This difference can be traced to polypseudologs having lost log character but polylogs retaining it. Also see the Appendix for the source of π.

V. CONCLUDING REMARKS

Riemann's ζ function of a negative number is not ordinary, being defined only through analytic continuation. This condition no doubt limits the number of possible avenues of approach to it. As far as we know, there is only one direct method of obtaining the values of $\zeta(n<0)$, given in the literature. It is by applying Cauchy's theorem of residues to Riemann's integral representation of Hurwitz's generalized ζ function [12,15]. Although not as general, our method of solution by comparison is elementary. It relies on two basic properties of polylogs—the recurrence and duplication relations. Polylogs have two independent parameters and special values of polylogs are obtained in certain limit processes peculiar to polylogs, one of which corresponds to Riemann's ζ function.

We have alluded that if $s_i \leqslant s_0 = 1$, the order of two possible limiting processes in polylogs ($s\to s_i$ and $z\to 1$) may not be exchanged. We can illustrate the difference through the following examples. Consider $\mathrm{Li}_s(z)$ taken in two different orders of s and z, when $s_i=1$ and 0:

$$\lim_{s\to 1}\lim_{z\to 1}\mathrm{Li}_s(z)=\zeta(s\to 1)=1/(s-1), \quad s\to 1 \tag{8a}$$

but

$$\lim_{z\to 1}\lim_{s\to 1}\mathrm{Li}_s(z)=-\log(1-z), \quad z\to 1. \tag{8b}$$

Also,

$$\lim_{s\to 0}\lim_{z\to 1}\mathrm{Li}_s(z)=\zeta(0)=-\tfrac{1}{2}, \tag{9a}$$

but

$$\lim_{z\to 1}\lim_{s\to 0}\mathrm{Li}_s(z)=1/(1-z), \quad z\to 1. \tag{9b}$$

In both cases we obtain quite different results depending on the order of limits taken. This difference extends to all lower order polylogs (i.e., polypseudologs) since $\mathrm{Li}_s(z)$, $s=n=-1,-2,\ldots$, all have poles of order $-n+1$ at $z=1$, whereas $\zeta(n)$ has been found finite. In the s-z plane, there is a line of singularity for $s=n\leqslant 1$ at $z=1$.

This difference in the limit process is important to the statistical thermodynamics, formulated in $\pm\mathrm{Li}_{d/2}(\pm z)$, respectively, for the massive Bose and Fermi gases, where z denotes the fugacity and d the number of physical dimensions. One finds in the Bose gas in lower dimensions a condition equivalent to taking the $d/2\to 1$ limit first. If $z\to 1$, then the divergence of $\mathrm{Li}_{d/2}(z=1)$, $d\leqslant 2$, implies that BEC does not occur in lower dimensions. But for the Fermi gas $z=1$ is not a singular point of $-\mathrm{Li}_{d/2}(-z)$. Hence the two limits may be exchanged harmlessly, indicating the regularness of the Fermi thermodynamics. In higher dimensions, where there are no divergent singular points, these gases at $z=1$ have a special significance, being a source of reflection symmetry. At this point the chemical potentials vanish and both gases (excluding kinematical factors) have the same universal entropy, a constant made up of Riemann's ζ function. When the chemical potential of a gas vanishes, it implies of course that the gas has no control over the flow of particles when in contact with a particle reservoir. These fundamental properties of the ideal gases are manifested in Riemann's ζ function, reached through the special limits of polylogs.

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APPENDIX : EVALUATION OF $\mathrm{Li}_n(-1)$ FROM THE INVERSION RELATION

There are only two general relations known for polylogs, the duplication and inversion formulas, the former given before [see (e)] and the latter to be given below. It is possible to obtain the values of $\mathrm{Li}_n(-1)$, $n=2,4,\ldots$ from the inversion formula, hence purely from properties of classical polylogs. To our knowledge, these values are ordinarily obtained from Riemann's ζ function through (a) and Eq. (2), hence indirectly. See p. 188 of Lewin's book [10].

The inversion relation is known—it is not difficult to establish it using the integral representation (1). Instead of writing down a general form (see Lewin's book [10] or Ref. [6]), it will suffice for our purpose to express it as follows:

$$\mathrm{Li}_{n+1}(-1/z)=(-1)^n\mathrm{Li}_{n+1}(-z)+(-1)^n a^{n+1}/\Gamma(n+2) + K_{n+1}(a), \quad n=1,2,\ldots \tag{A1}$$

where $a=\log z$ and K_{n+1} is a polynomial in a in which $\mathrm{Li}_n(-1)$, $\mathrm{Li}_{n-2}(-1)$, $\ldots$ are its coefficients. The first few are listed below:

$$K_2=2\mathrm{Li}_2(-1), \tag{A2a}$$

$$K_3=-2a\mathrm{Li}_2(-1), \tag{A2b}$$

$$K_4=2[\mathrm{Li}_4(-1)+(a^2/2)\mathrm{Li}_2(-1)], \tag{A2c}$$

$$K_5=-2[a\mathrm{Li}_4(-1)+(a^3/3!)\mathrm{Li}_2(-1)], \tag{A2d}$$

$$K_6 = 2[\mathrm{Li}_6(-1) + (a^2/2)\mathrm{Li}_4(-1) + (a^4/4!)\mathrm{Li}_2(-1)], \quad \text{(A2e)}$$

$$K_7 = -2[a\mathrm{Li}_6(-1) + (a^3/3!)\mathrm{Li}_4(-1) + (a^5/5!)\mathrm{Li}_2(-1)], \quad \text{(A2f)}$$

etc. Observe that if $z=-1$ *and* $n=2,4,6,\ldots$ (i.e., even numbers), Eq. (A1) becomes

$$a^{n+1}/\Gamma(n+2) + K_{n+1}(a) = 0, \quad \text{(A3)}$$

where now $a(z=-1)=i\pi$. If $n=2$, we have $a^3/3!+K_3 = a^3/3! - 2a\mathrm{Li}_2(-1)=0$. Hence $\mathrm{Li}_2(-1)=-\pi^2/12$ and $\zeta(2)=\pi^2/6$ by Eq. (2).

If $n=4$, we have $a^5/5!+K_5=0$. Hence using Eq. (A2d) we find that $\mathrm{Li}_4(-1)=-7\pi^4/6!$, recovering $\zeta(4)=\pi^4/90$. Similarly if $n=6$, we obtain $\mathrm{Li}_6(-1)=-31\pi^6/7!6$, recovering $\zeta(6)=\pi^6/945$. In this way it is possible to obtain any desired values of $\mathrm{Li}_n(-1)$ and hence also $\zeta(n)$, $n=2,4,6,\ldots$ directly from the two general properties of polylogs. Also observe that the source of π in $\zeta(n)$, $n=2,4,6,\ldots$ is log($z=-1$). It is interesting to note that we can obtain in this way Riemann's ζ function of only positive even integral order.

In the classical theory of polylogs the values of $\mathrm{Li}_{2n}(-1)$, $n=1,2,\ldots$, are also obtained by combining the inversion relation with the duplication relation. See pp. 172 and 173 of Lewin's book [10]. As a result, the ease and transparency of our approach are not easily seen, which is based on the inversion property alone.

Perhaps the simplest method of obtaining the values of $\zeta(2n)$, $n=1,2,\ldots$ is the following one, evidently due to Euler [16]. Since the zeros of $\sin x$ are $x=\pm k\pi$, $k=0,1,2,\ldots$ one can write

$$\sin x = \sum_{m=0}^{\infty} (-)^m x^{2m+1}/(2m+1)! = x\prod_{k=1}^{\infty}\left[1-\left(\frac{x}{k\pi}\right)^2\right]. \quad \text{(A4)}$$

If the coefficients of x^{2m+1}, $m\geqslant 1$ are now equated, the values of $\zeta(2m)$ follow immediately. It is certainly much simpler than the standard method of obtaining them by contour integration [17]. More generally one can obtain $\zeta(n)$, $n>1$, by Riemann's self-reflection formula since $\zeta(1-n)$ can be evaluated through the Hurwitz function [12,15].

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